Math 8 Homework 6

1 Clever Counting

(a) How many ordered triples (A, B, C) of sets satisfy both $A \cap B \cap C = \varnothing$ and $A \cup B \cup C = \{1, 2, \ldots, n\}$?

(b) Given $n \in \mathbb{N}$, prove that $x = (n^2)!/(n!)^n$ is an integer by inventing a counting problem for which x is the answer.

(c) Let S be a set with n elements. Evaluate each of these in closed form:

(i) $\sum_{A \subseteq S} |A|$

(ii) $\sum_{A, B \subseteq S} |A \cap B|$

(d) Given positive integers n, k with $k \leq n$, the Stirling number of the second kind $S(n, k)$ is defined to be the number of ways to place n balls into k identical boxes, leaving no box empty.

(i) Evaluate $S(n, 1), S(n, 2)$, $S(n, n-1)$ and $S(n, n)$.

(ii) Count two ways to prove that

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

(iii) Use the previous parts and induction to prove that

$$S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$$

(e) Let S be a set of n real numbers. Given each nonempty subset of S we can compute its average; in this way we can produce $2^n - 1$ numbers. Prove that the average of these $2^n - 1$ numbers is the average of S.

(f) Let $S = \{1, 2, \ldots, n\}$. Each k–element subset of S has a smallest element; in this way we produce $\binom{n}{k}$ numbers. Prove the average of these smallest elements is $(n+1)/(k+1)$.

2 Binomial Identities

For each of the following, invent a situation and count in two ways to prove the identity. Assume all variables are arbitrary positive integers.

(a) $\binom{n}{k} = \binom{n}{n-k}$

(b) $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$

(c) $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$

(d) $\sum_k \binom{n}{k} = 2^n$

(e) $\sum_k k\binom{n}{k} = n2^{n-1}$

(f) $\sum_k \binom{n}{k}\binom{m}{r-k} = \binom{n+m}{r}$

(g) $\sum_k k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$